REAL ANALYTICITY OF SOLUTIONS TO SCHRÖDINGER EQUATIONS INVOLVING A FRACTIONAL LAPLACIAN AND OTHER FOURIER MULTIPLIERS

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Abstract. We prove analyticity of solutions to certain nonlocal linear Schrödinger equations with analytic potentials.

1. Introduction, result, and proof.

In [1], we proved the real analyticity away from the Coulomb singularity of atomic pseudorelativistic Hartree-Fock orbitals. The proof works for solutions to a variety of equations (see [1, Remark 1.2]), in particular, any $H^{1/2}$-solution $\varphi : \mathbb{R}^3 \to \mathbb{C}$ to the non-linear equation

$$(\sqrt{-\Delta+1})\varphi - \frac{Z}{|\cdot|}\varphi \pm \big(|\varphi|^2 * |\cdot|^{-1}\big)\varphi = \lambda\varphi \tag{1}$$

is real analytic away from $\mathbf{x} = 0$. The emphasis in [1] was on the Coulomb singularity $|\cdot|^{-1}$ and on the Hartree-term $(|\varphi|^2 * |\cdot|^{-1})\varphi$. However, the result holds for much more general potentials V than $|\cdot|^{-1}$. We state and prove this in the linear case here (refering to [1] for certain technical points of the proof).

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be an open set, and assume $V : \mathbb{R}^3 \to \mathbb{C}$ is real analytic in Ω, that is, $V \in C^\omega(\Omega)$. Let $s \in [1/2, 1]$, $m > 0$, or $s = 1/2, m = 0$, and assume $\varphi \in H^{2s}(\mathbb{R}^3)$ is a solution to*

$$E_{s,m}(\mathbf{p})\varphi := \big(-\Delta + m\big)^s\varphi = V\varphi \quad in \quad \mathbb{R}^3\,. \tag{2}$$

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Assume furthermore that $V \in L^t(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ *with*

$$\begin{cases} t = 3/4s & \text{if} \quad s \in [1/2, 3/4)\,, \\ t > 1 & \text{if} \quad s = 3/4\,, \\ t = 1 & \text{if} \quad s \in (3/4, 1)\,. \end{cases} \tag{3}$$

Then $\varphi \in C^\omega(\Omega)$, *that is,* φ *is real analytic in* Ω.

Remark 1.2. In the case $s = 1$, the result is well-known, and no integrability condition on V is needed, the equation being local in this case. The integrability conditions on V seem unnecessary, but are needed for our method to work (see (27) and after). Note that if $V \in L^p(\mathbb{R}^3)$ for some $p \in [1, \infty)$, then $V \in L^q(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ for all $q \in [1, p]$. As in [1] our proof is based on the classical proof by Morrey and Nirenberg (see [2]). In order to deal with the non-locality we use the localization result in Lemma A.1 and the analytic smoothing estimate in Lemma A.2 in the Appendix below (for more details see [1, after Remark 1.4]).

To prove Theorem 1.1, it suffices (using Sobolev embedding) to prove the following proposition (for details, see [1, after Proposition 2.1]. Note that in the linear case, it suffices to work in $L^2(\mathbb{R}^3)$.)

Proposition 1.3. *Let the assumptions be as in Theorem 1.1. Let* $\mathbf{x}_0 \in \Omega$, $R = \min\{1, \operatorname{dist}(\mathbf{x}_0, \Omega^c)/4\}$, *and* $\omega = B_R(\mathbf{x}_0)$ $(\subset\subset \Omega)$. *Define* $\omega_\delta = B_{R-\delta}(\mathbf{x}_0)$ *for* $\delta > 0$.

Then there exist constants $C, B > 1$ *such that for all* $j \in \mathbb{N}$, *and for all* $\epsilon > 0$ *such that* $\epsilon j \le R/2$, *we have*

$$\epsilon^{|\beta|} \|D^\beta \varphi\|_{L^2(\omega_{\epsilon j})} \le C B^{|\beta|} \quad \textit{for all} \quad \beta \in \mathbb{N}_0^3 \quad \textit{with} \quad |\beta| \le j\,. \tag{4}$$

Proof. This is by induction. For $j \in \mathbb{N}_0$ (and constants $C, B > 1$ to be determined below), let $\mathcal{P}(j)$ be the statement: For all $\epsilon > 0$ with $\epsilon j \le R/2$ we have

$$\epsilon^{|\beta|} \|D^\beta \varphi\|_{L^2(\omega_{\epsilon j})} \le C\, B^{|\beta|} \quad \text{for all} \quad \beta \in \mathbb{N}_0^3 \quad \text{with} \quad |\beta| \le j\,. \tag{5}$$

Choosing $C \ge \|\varphi\|_{H^1(\mathbb{R}^3)}$ and $B > 1$ ensures that both $\mathcal{P}(0)$ and $\mathcal{P}(1)$ hold (since $s \in [1/2, 1)$, and $\epsilon \le R/2 \le 1$ for $j = 1$). The induction hypothesis is: Let $j \in \mathbb{N}$, $j \ge 1$. Then $\mathcal{P}(\tilde{j})$ holds for all $\tilde{j} \le j$. We now prove that $\mathcal{P}(j+1)$ holds. By the definition of ω_δ and the induction hypothesis, it suffices to study $\beta \in \mathbb{N}_0^3$ with $|\beta| = j + 1$. It therefore remains to prove that

$$\epsilon^{|\beta|} \|D^\beta \varphi\|_{L^2(\omega_{\epsilon(j+1)})} \le C\, B^{|\beta|} \quad \text{for all} \quad \epsilon > 0 \quad \text{with} \quad \epsilon(j+1) \le R/2 \\ \text{and all} \quad \beta \in \mathbb{N}_0^3 \quad \text{with} \quad |\beta| = j + 1\,. \tag{6}$$

Let ϵ and β be as in (6). It is convenient to write, for $\ell > 0$, $\epsilon > 0$ such that $\epsilon\ell \le R/2$, and $\sigma \in \mathbb{N}_0^3$ with $0 < |\sigma| \le j$,

$$\|D^\sigma \varphi\|_{L^2(\omega_{\epsilon\ell})} = \|D^\sigma \varphi\|_{L^2(\omega_{\tilde{\epsilon}\tilde{j}})} \quad \text{with} \quad \tilde{\epsilon} = \frac{\epsilon\ell}{|\sigma|},\ \tilde{j} = |\sigma|\,,$$

so that, by the induction hypothesis (applied on the term with $\tilde{\epsilon}$ and $\tilde{j}$) we get that

$$\|D^\sigma \varphi\|_{L^2(\omega_{\epsilon\ell})} \le C\Big(\frac{B}{\tilde{\epsilon}}\Big)^{|\sigma|} = C\Big(\frac{|\sigma|}{\ell}\Big)^{|\sigma|}\Big(\frac{B}{\epsilon}\Big)^{|\sigma|}\,. \tag{7}$$

Compare this with (5). With the convention that $0^0 = 1$, (7) also holds for $|\sigma| = 0$.

Inverting the equation (2) when $m > 0$, we have (in $L^2(\mathbb{R}^3)$)

$$\varphi = E_{s,m}(\mathbf{p})^{-1} V\varphi\,. \tag{8}$$

For the case $s = 1/2, m = 0$,

$$\varphi = \big[(-\Delta)^{1/2} + 1\big]^{-1}\widetilde{V}\varphi =: \widetilde{E}_{1/2,0}(\mathbf{p})^{-1}\widetilde{V}\varphi\,, \tag{9}$$

with $\widetilde{V} = V + 1 \in L^t(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$. Note that $1 \in C^\omega(\mathbb{R}^3)$.

We choose a function Φ (depending on j) satisfying

$$\Phi \in C_0^\infty(\omega_{\epsilon(j+3/4)})\,, \quad 0 \le \Phi \le 1\,, \quad \text{with } \Phi \equiv 1 \text{ on } \omega_{\epsilon(j+1)}\,. \tag{10}$$

Then $\|D^\beta\varphi\|_{L^2(\omega_{\epsilon(j+1)})} \le \|\Phi D^\beta \varphi\|_{L^2(\mathbb{R}^3)} =: \|\Phi D^\beta\varphi\|_2$. The estimate (6)–and hence, by induction, the proof of Proposition 1.3—now follows from (8) and (9) and the following lemma. □

Lemma 1.4. *Assume the induction hypothesis described above holds. Let Φ be as in* (10)*. Then for all $\epsilon > 0$ with $\epsilon(j+1) \le R/2$, and all $\beta \in \mathbb{N}_0^3$ with $|\beta| = j+1$, $\Phi D^\beta E_{s,m}(\mathbf{p})^{-1}V\varphi$ belongs to $L^2(\mathbb{R}^3)$, and*

$$\|\Phi D^\beta E_{s,m}(\mathbf{p})^{-1} V\varphi\|_2 \le C\Big(\frac{B}{\epsilon}\Big)^{|\beta|}\,, \tag{11}$$

where $C, B > 1$ are the constants in (5)*. The same holds for $\Phi\widetilde{E}_{1/2,0}(\mathbf{p})^{-1}\widetilde{V}\varphi$.*

Proof. Let $\sigma \in \mathbb{N}_0^3$ and $\nu \in \{1,2,3\}$ be such that $\beta = \sigma + e_\nu$, so that $D^\beta = D_\nu D^\sigma$. Notice that $|\sigma| = j$. Choose localization functions $\{\chi_k\}_{k=0}^j$ and $\{\eta_k\}_{k=0}^j$ as described in the Appendix below. Since $V\varphi \in L^2(\mathbb{R}^3)$, and $E_{s,m}(\mathbf{p})^{-1}$ maps $H^r(\mathbb{R}^3)$ to $H^{r+2s}(\mathbb{R}^3)$ for all $r \in \mathbb{R}$,

Lemma A.1 below (with $\ell = j$) implies that

$$\begin{aligned}\Phi D^{\beta} E_{s,m}(\mathbf{p})^{-1}[V\varphi] = &\sum_{k=0}^{j} \Phi D_{\nu} E_{s,m}(\mathbf{p})^{-1} D^{\beta_k}\chi_k D^{\sigma-\beta_k}[V\varphi] \\ &+ \sum_{k=0}^{j-1} \Phi D_{\nu} E_{s,m}(\mathbf{p})^{-1} D^{\beta_k}[\eta_k, D^{\mu_k}] D^{\sigma-\beta_{k+1}}[V\varphi] \\ &+ \Phi D_{\nu} E_{s,m}(\mathbf{p})^{-1} D^{\sigma}[\eta_j V\varphi]\,, \end{aligned} \tag{12}$$

as an identity in $H^{-|\beta|+2s}(\mathbb{R}^3)$. Similarly for $\widetilde{E}_{1/2,0}(\mathbf{p})^{-1}$. Here, $[\,\cdot\,,\,\cdot\,]$ denotes the commutator. Also, $|\beta_k| = k$, $|\mu_k| = 1$, and $0 \le \eta_k, \chi_k \le 1$. (For the support properties of η_k, χ_k, see the Appendix.) We will prove that each term on the right side of (12) belong to $L^2(\mathbb{R}^3)$, and bound their norms. The proof of (11) will follow by summing these bounds.

The first sum in (12). Let θ_k be the characteristic function of the support of χ_k (which is contained in ω). We can estimate, for $k \in \{0, \dots, j\}$,

$$\begin{aligned}&\|\Phi D_{\nu} E_{s,m}(\mathbf{p})^{-1} D^{\beta_k}\chi_k D^{\sigma-\beta_k}[V\varphi]\|_2 \\ &\qquad = \|(\Phi E_{s,m}(\mathbf{p})^{-1} D_{\nu} D^{\beta_k}\chi_k)\theta_k D^{\sigma-\beta_k}[V\varphi]\|_2 \\ &\qquad \le \|\Phi E_{s,m}(\mathbf{p})^{-1} D_{\nu} D^{\beta_k}\chi_k\|_{\mathcal{B}}\, \|\theta_k D^{\sigma-\beta_k}[V\varphi]\|_2\,. \end{aligned} \tag{13}$$

Here, $\|\cdot\|_{\mathcal{B}}$ is the operator norm on the bounded operators on $L^2(\mathbb{R}^3)$.

For $k = 0$, the first factor on the right side of (13) can be estimated using the spectral theorem, since $s \in [1/2, 1)$. This way, since $\|\chi_0\|_\infty = \|\Phi\|_\infty = 1$,

$$\|\Phi E_{s,m}(\mathbf{p})^{-1} D_{\nu}\chi_0\|_{\mathcal{B}} \le C_s(m)\,. \tag{14}$$

This also holds for $\widetilde{E}_{1/2,0}(\mathbf{p})^{-1}$.

For $k > 0$, the first factor on the right side of (13) can (also for $\widetilde{E}_{1/2,0}(\mathbf{p})^{-1}$) be estimated using (38) in Lemma A.2 below (with $\mathfrak{r} = 1$, $\mathfrak{q} = \mathfrak{q}^* = \mathfrak{p} = 2$). Since

$$\operatorname{dist}(\operatorname{supp}\chi_k, \operatorname{supp}\Phi) \ge \epsilon(k - 1 + 1/4)$$

and $\|\chi_k\|_\infty = \|\Phi\|_\infty = 1$, this gives (since $(\beta_k + e_\nu)! \le (|\beta_k| + 1)! = (k+1)!$) that

$$\begin{aligned}&\|\Phi E_{s,m}(\mathbf{p})^{-1} D_{\nu} D^{\beta_k}\chi_k\|_{\mathcal{B}} \\ &\qquad \le c_s \frac{(k+1)!}{k+1-2s}\Big(\frac{8}{\epsilon(k-1+1/4)}\Big)^{k+1}\big[\epsilon(k-1+1/4)\big]^{2s}\,.\end{aligned}$$

Since $s \in [1/2, 1)$, and $\epsilon(k-1+1/4) \le \epsilon(j+1) \le R/2 \le 1$, this implies

$$\|\Phi E_{s,m}(\mathbf{p})^{-1} D_\nu D^{\beta_k}\chi_k\|_{\mathcal{B}} \le c_s \frac{32}{2-2s}\Big(\frac{8}{\epsilon}\Big)^k = \tilde{c}_s\Big(\frac{8}{\epsilon}\Big)^k . \tag{15}$$

For $s \in (0, 1/2)$ one does not gain the needed power of ϵ here.

It follows from (14) and (15) that, for all $k \in \{0, \dots, j\}$, $\nu \in \{1, 2, 3\}$,

$$\|\Phi E_{s,m}(\mathbf{p})^{-1} D_\nu D^{\beta_k}\chi_k\|_{\mathcal{B}} \le \widetilde{C}_s(m)\Big(\frac{8}{\epsilon}\Big)^k , \tag{16}$$

with $\widetilde{C}_s(m) := \tilde{c}_s + C_s(m)$.

It remains to estimate the second factor in (13). For this, we employ the analyticity of V. Let $A = A(\mathbf{x}_0) \ge 1$ be such that, for all $\sigma \in \mathbb{N}_0^3$,

$$\sup_{\mathbf{x}\in\omega} |D^\sigma V(\mathbf{x})| \le A^{|\sigma|+1}|\sigma|! \,. \tag{17}$$

The existence of A follows from the real analyticity in $\omega = B_R(\mathbf{x}_0) \subset\subset \Omega$ of V (see e. g. [3, Proposition 2.2.10]). It follows (since $\omega_\delta = \emptyset$ for $\delta \ge 1$) that, for all $\epsilon > 0$, $\ell \in \mathbb{N}_0$, and $\sigma \in \mathbb{N}_0^3$,

$$\epsilon^{|\sigma|} \sup_{\mathbf{x}\in\omega_{\epsilon\ell}} |D^\sigma V(\mathbf{x})| \le A^{|\sigma|+1}|\sigma|! \, \ell^{-|\sigma|} \,, \tag{18}$$

with $\omega_{\epsilon\ell} \subseteq \omega$ as in defined in Proposition 1.3.

For $k = j$, since $\beta_j = \sigma$, we find, by (18) and the choice of C, that

$$\|\theta_j V\varphi\|_2 \le \|V\|_{L^\infty(\omega)}\|\varphi\|_{L^2(\omega)} \le CA \,. \tag{19}$$

For $k \in \{0, \dots, j-1\}$ we get, by Leibniz's rule, that

$$\begin{aligned} &\|\theta_k D^{\sigma-\beta_k}[V\varphi]\|_2 \\ &\qquad \le \sum_{\mu\le\sigma-\beta_k} \binom{\sigma-\beta_k}{\mu} \|\theta_k D^\mu V\|_\infty \, \|\theta_k D^{\sigma-\beta_k-\mu}\varphi\|_2 \,. \end{aligned} \tag{20}$$

Now, $\operatorname{supp}\theta_k = \operatorname{supp}\chi_k \subseteq \omega_{\epsilon(j-k+1/4)}$, so by (18), for all $\mu \le \sigma - \beta_k$,

$$\|\theta_k D^\mu V\|_\infty \le \sup_{\mathbf{x}\in\omega_{\epsilon(j-k+1/4)}} |D^\mu V(\mathbf{x})| \le \epsilon^{-|\mu|}A^{|\mu|+1}|\mu|!(j-k)^{-|\mu|} \,. \tag{21}$$

By the induction hypothesis (in (7)),

$$\begin{aligned} \|\theta_k D^{\sigma-\beta_k-\mu}\varphi\|_2 &\le \|D^{\sigma-\beta_k-\mu}\varphi\|_{L^2(\omega_{\epsilon(j-k)})} \\ &\le C\Big(\frac{|\sigma-\beta_k-\mu|}{j-k}\Big)^{|\sigma-\beta_k-\mu|}\Big(\frac{B}{\epsilon}\Big)^{|\sigma-\beta_k-\mu|} . \end{aligned} \tag{22}$$

It follows from (20), (21), and (22) (using that $|\sigma| = j, |\beta_k| = k$, so $\sum_{\mu\le\sigma-\beta_k,|\mu|=m}\binom{\sigma-\beta_k}{\mu} = \binom{|\sigma-\beta_k|}{m} = \binom{j-k}{m}$, and then summing over m)

that

$$\|\theta_k D^{\sigma-\beta_k}[V\varphi]\|_2 \leq CA\Big(\frac{B}{\epsilon}\Big)^{j-k}\sum_{m=0}^{j-k}\binom{j-k}{m}\frac{m!(j-k-m)^{j-k-m}}{(j-k)^{j-k}}\Big(\frac{A}{B}\Big)^m. \tag{23}$$

As in [1, (62)], this implies (choosing $B > 2A$), that, for any $k \in \{0,\dots,j-1\}$,

$$\|\theta_k D^{\sigma-\beta_k}[V\varphi]\|_2 \leq CA\Big(\frac{B}{\epsilon}\Big)^{j-k}\sum_{m=0}^{j-k}\Big(\frac{A}{B}\Big)^m \leq 2CA\Big(\frac{B}{\epsilon}\Big)^{j-k}. \tag{24}$$

Note that, by (19), the same estimate holds true if $k = j$.

So, from (13), (16), (24), the fact that $\epsilon \leq 1$ (since $\epsilon(j+1) \leq R/2 \leq 1/2$), and choosing $B > 16, B > 12A\widetilde{C}_s(m)$, it follows that (also for $\widetilde{E}_{1/2,0}(\mathbf{p})^{-1}$)

$$\Big\|\sum_{k=0}^{j}\Phi D_\nu E_{s,m}(\mathbf{p})^{-1}D^{\beta_k}\chi_k D^{\sigma-\beta_k}[V\varphi]\Big\|_2 \tag{25}$$
$$\leq 2CA\widetilde{C}_s(m)\Big(\frac{B}{\epsilon}\Big)^j\sum_{k=0}^{j}\Big(\frac{8}{B}\Big)^k \leq C(4A\widetilde{C}_s(m))\Big(\frac{B}{\epsilon}\Big)^j \leq \frac{C}{3}\Big(\frac{B}{\epsilon}\Big)^{j+1}.$$

The second sum in (12). Note that $[\eta_k, D^{\mu_k}] = -(D^{\mu_k}\eta_k)$ (recall that $|\mu_k| = 1$). The second sum in (12) is the first one with j replaced by $j-1$ and χ_k replaced by $-D^{\mu_k}\eta_k$. The operator $D^{\sigma-\beta_{k+1}}$ contains $(j-1)-k$ derivatives instead of the $j-k$ in $D^{\sigma-\beta_k}$. Then, to control $D^{\sigma-\beta_{k+1}}[V\varphi_i]$ (with the same method used above for $D^{\sigma-\beta_k}[V\varphi_i]$) we need that $\operatorname{supp} D^{\mu_k}\eta_k$ is contained in $\omega_{\epsilon((j-1)-k+1/4)}$. We have more: as for χ_k we have $\operatorname{supp} D^{\mu_k}\eta_k \subseteq \omega_{\epsilon(j-k+1/4)} \subseteq \omega_{\epsilon((j-1)-k+1/4)}$. Finally, $\|D^{\mu_k}\eta_k\|_\infty \leq C_*/\epsilon$, with $C_* > 0$ the constant in (34) in the Appendix.

It follows that the second sum in (12) can be estimated as the first one, up to *one* extra factor of C_*/ϵ *and* up to replacing j by $j-1$ in the estimate (25). Hence, using that $\epsilon \leq 1$, the choice of B above, and choosing $B \geq C_*$, we get that

$$\Big\|\sum_{k=0}^{j-1}\Phi D_\nu E_{s,m}(\mathbf{p})^{-1}D^{\beta_k}[\eta_k, D^{\mu_k}]D^{\sigma-\beta_{k+1}}[V\varphi]\Big\|_2 \tag{26}$$
$$\leq \frac{C_*}{\epsilon}C(4A\widetilde{C}_s(m))\Big(\frac{B}{\epsilon}\Big)^{j-1} \leq C(4A\widetilde{C}_s(m))\Big(\frac{B}{\epsilon}\Big)^j \leq \frac{C}{3}\Big(\frac{B}{\epsilon}\Big)^{j+1}.$$

The last term in (12). It remains to study

$$\Phi D^\beta E_{s,m}(\mathbf{p})^{-1}[\eta_j V\varphi] \ \text{ and } \ \Phi D^\beta \widetilde{E}_{1/2,0}(\mathbf{p})^{-1}[\eta_j \widetilde{V}\varphi]\,. \tag{27}$$

Recall that Φ is supported in $\omega_{\epsilon(j+1)}$ and (see Appendix)

$$\mathrm{dist}(\mathrm{supp}\,\Phi, \mathrm{supp}\,\eta_j) \geq \epsilon(j+1/4)\,. \tag{28}$$

Recall that $V \in L^t(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ for some t (see (3)). Again, we use Lemma A.2, this time with $\mathfrak{q} = \mathfrak{q}^* = 2$, and $\mathfrak{p} = 2$, $\mathfrak{r} = 1$ (for the L^∞-part), and $\mathfrak{p} = 6/(3+4s)$, $\mathfrak{r} = 3/(3-2s)$ (for the L^t-part).

Then $\mathfrak{p}^{-1} + \mathfrak{q}^{-1} + \mathfrak{r}^{-1} = 2$, $\mathfrak{p} \in [1,\infty)$, $\mathfrak{q} > 1$, $\mathfrak{r} \in [1,\infty)$, and $\mathfrak{q}^{-1} + \mathfrak{q}^{*-1} = 1$. This gives that

$$\begin{aligned}\|\Phi D^\beta E_{s,m}(\mathbf{p})^{-1}[\eta_j V\varphi]\|_2 &\leq \|\Phi D^\beta E_{s,m}(\mathbf{p})^{-1}\eta_j\|_{\mathcal{B}_{\mathfrak{p},2}}\|V\varphi\|_{\mathfrak{p}} \\ &\leq c_s(\mathfrak{r})\beta!\Big(\frac{8}{\epsilon(j+1/4)}\Big)^{|\beta|}\big(\epsilon(j+1/4)\big)^{3/\mathfrak{r}-3+2s}\times \\ &\qquad\qquad \times \big[\mathfrak{r}(|\beta|+3-2s)-3\big]^{-1/\mathfrak{r}}\|V\varphi\|_{\mathfrak{p}}\,.\end{aligned}$$

As before, we used that $\|\Phi\|_\infty = \|\eta_j\|_\infty = 1$. The same estimate holds for $\widetilde{E}_{1/2,0}(\mathbf{p})^{-1}$. Note that

$$\beta!\Big(\frac{8}{j+1/4}\Big)^{|\beta|} \leq 32^{|\beta|}\frac{|\beta|!}{(j+1)^{|\beta|}} = 32^{|\beta|}\frac{(j+1)!}{(j+1)^{j+1}} \leq 32^{|\beta|}\,. \tag{29}$$

Since $\epsilon(j+1) \leq R/2 < 1$ and $3/\mathfrak{r}-3+2s \geq 0$ (in both cases), it follows that $(\epsilon(j+1/4))^{3/\mathfrak{r}-3+2s} \leq 1$. Also, since $|\beta| = j+1 \geq 2$, $\mathfrak{r} \geq 1$, we have $\mathfrak{r}(|\beta|+3-2s)-3 \geq 2-2s > 0$, hence $\big[\mathfrak{r}(|\beta|+3-2s)-3\big]^{-1/\mathfrak{r}} \leq (2-2s)^{-1/\mathfrak{r}}$. It follows that (also for $\widetilde{E}_{1/2,0}(\mathbf{p})^{-1}$)

$$\|\Phi D^\beta E_{s,m}(\mathbf{p})^{-1}\eta_j V\|_2 \leq \frac{c_s(\mathfrak{r})}{(2-2s)^{1/\mathfrak{r}}}\Big(\frac{32}{\epsilon}\Big)^{|\beta|}\|V\varphi\|_{\mathfrak{p}}\,. \tag{30}$$

It remains to note that, using the stated conditions on t (see (3)), Sobolev embedding (for $\varphi \in H^{2s}(\mathbb{R}^3)$), and Hölder's inequality, one has (in all cases), $\|V\varphi\|_{\mathfrak{p}} < \infty$, for the stated choices of $\mathfrak{p}$. Hence, choosing $B > 32$ and $C \geq 3c_s(\mathfrak{r})(2-2s)^{-1/\mathfrak{r}}\|V\varphi\|_{\mathfrak{p}}$ (recall that $|\beta| = j+1$),

$$\|\Phi D^\beta E_{s,m}(\mathbf{p})^{-1}\eta_j V\|_2 \leq \frac{C}{3}\Big(\frac{B}{\epsilon}\Big)^{j+1}\,. \tag{31}$$

The same holds for $\widetilde{E}_{1/2,0}(\mathbf{p})^{-1}$.

The estimate (11) now follows from (12) and the estimates (25), (26), and (31). Note: This argument ('*The last term in* (12)') works for $E_{s,m}(\mathbf{p})^{-1}$ for all $s \in (0,1), m > 0$ (with the same condition on V when $s \in (0,1/2)$ as for $s \in [1/2, 3/4)$).

□

Appendix A. Lemmata from [1]

Recall (see (10)) that we have chosen a function Φ (depending on j) satisfying

$$\Phi \in C_0^\infty(\omega_{\epsilon(j+3/4)}) \,, \quad 0 \le \Phi \le 1 \,, \quad \text{with } \Phi \equiv 1 \text{ on } \omega_{\epsilon(j+1)} \,. \tag{32}$$

For $j \in \mathbb{N}$ we choose functions $\{\chi_k\}_{k=0}^j$, and $\{\eta_k\}_{k=0}^j$ (all depending on j) with the following properties (for an illustration, see [1, Figures 1 and 2]). The functions $\{\chi_k\}_{k=0}^j$ are such that

$$\chi_0 \in C_0^\infty(\omega_{\epsilon(j+1/4)}) \quad \text{with} \quad \chi_0 \equiv 1 \quad \text{on } \omega_{\epsilon(j+1/2)} \,,$$

and, for $k = 1, \ldots, j$,

$$\begin{aligned} &\chi_k \in C_0^\infty(\omega_{\epsilon(j-k+1/4)}) \\ &\qquad \text{with} \begin{cases} \chi_k \equiv 1 & \text{on } \omega_{\epsilon(j-k+1/2)} \setminus \omega_{\epsilon(j-k+1+1/4)} \,, \\ \chi_k \equiv 0 & \text{on } \mathbb{R}^3 \setminus \big(\omega_{\epsilon(j-k+1/4)} \setminus \omega_{\epsilon(j-k+1+1/2)}\big) \,. \end{cases} \end{aligned}$$

Finally, the functions $\{\eta_k\}_{k=0}^j$ are such that for $k = 0, \ldots, j$,

$$\eta_k \in C^\infty(\mathbb{R}^3) \quad \text{with} \quad \begin{cases} \eta_k \equiv 1 & \text{on } \mathbb{R}^3 \setminus \omega_{\epsilon(j-k+1/4)} \,, \\ \eta_k \equiv 0 & \text{on } \omega_{\epsilon(j-k+1/2)} \,. \end{cases}$$

Moreover we ask that

$$\begin{aligned} &\chi_0 + \eta_0 \equiv 1 && \text{on } \mathbb{R}^3, \\ &\chi_k + \eta_k \equiv 1 && \text{on } \mathbb{R}^3 \setminus \omega_{\epsilon(j-k+1+1/4)} \text{ for } k = 1, \ldots, j \,, \\ &\eta_k \equiv \chi_{k+1} + \eta_{k+1} && \text{on } \mathbb{R}^3 \text{ for } k = 0, \ldots, j-1 \,. \end{aligned} \tag{33}$$

Lastly, we choose these localization functions such that, for a constant $C_* > 0$ (independent of ϵ, k, j, β) and for all $\beta \in \mathbb{N}_0^3$ with $|\beta| = 1$, we have that

$$|D^\beta \chi_k(\mathbf{x})| \le \frac{C_*}{\epsilon} \quad \text{and} \quad |D^\beta \eta_k(\mathbf{x})| \le \frac{C_*}{\epsilon} \,, \tag{34}$$

for $k = 0, \ldots, j$, and all $\mathbf{x} \in \mathbb{R}^3$.

The next lemma shows how to use these localization functions.

Lemma A.1. *For $j \in \mathbb{N}$ fixed, choose functions $\{\chi_k\}_{k=0}^j$, and $\{\eta_k\}_{k=0}^j$ as above, and let $\sigma \in \mathbb{N}_0^3$ with $|\sigma| = j$. For $\ell \in \mathbb{N}$ with $\ell \le j$, choose multiindices $\{\beta_k\}_{k=0}^\ell$ such that:*

$$|\beta_k| = k \text{ for } k = 0, \ldots, \ell \,, \ \beta_{k-1} < \beta_k \text{ for } k = 1, \ldots, \ell, \text{ and } \beta_\ell \le \sigma \,.$$

Then for all $g \in \mathcal{S}'(\mathbb{R}^3)$,

$$\begin{aligned} D^\sigma g &= \sum_{k=0}^{\ell} D^{\beta_k} \chi_k D^{\sigma-\beta_k} g \\ &\quad + \sum_{k=0}^{\ell-1} D^{\beta_k}[\eta_k, D^{\mu_k}] D^{\sigma-\beta_{k+1}} g + D^{\beta_\ell}\eta_\ell D^{\sigma-\beta_\ell} g \,, \end{aligned} \tag{35}$$

with $\mu_k = \beta_{k+1} - \beta_k$ for $k = 0, \dots, \ell-1$ (hence, $|\mu_k| = 1$).

For a proof, see [1, Lemma B.1].

For $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$, denote by $\| \cdot \|_{\mathcal{B}_{\mathfrak{p},\mathfrak{q}}}$ the operator norm on bounded operators from $L^{\mathfrak{p}}(\mathbb{R}^3)$ to $L^{\mathfrak{q}}(\mathbb{R}^3)$.

Lemma A.2. *For $s \in (0,1)$, $m > 0$, let $E_{s,m}(\mathbf{p})^{-1} = (-\Delta + m)^{-s}$. For all $\mathfrak{p}, \mathfrak{r} \in [1, \infty)$, $\mathfrak{q} \in (1, \infty)$, with $\mathfrak{p}^{-1} + \mathfrak{q}^{-1} + \mathfrak{r}^{-1} = 2$, all $\beta \in \mathbb{N}_0^3$ (with $|\beta| > 1$ if $\mathfrak{r} = 1$), and all $\Phi, \chi \in C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ with*

$$\operatorname{dist}(\operatorname{supp}(\chi), \operatorname{supp}(\Phi)) \geq d \,, \tag{36}$$

the operator $\Phi E_{s,m}(\mathbf{p})^{-1} D^\beta \chi$ is bounded from $L^{\mathfrak{p}}(\mathbb{R}^3)$ to $(L^{\mathfrak{q}}(\mathbb{R}^3))' = L^{\mathfrak{q}^*}(\mathbb{R}^3)$ (with $\mathfrak{q}^{-1} + \mathfrak{q}^{*-1} = 1$), and

$$\begin{aligned} &\|\Phi E_{s,m}(\mathbf{p})^{-1} D^\beta \chi\|_{\mathcal{B}_{\mathfrak{p},\mathfrak{q}^*}} \\ &\quad \leq c_s(\mathfrak{r}) \beta! \Big(\frac{8}{d}\Big)^{|\beta|} d^{3/\mathfrak{r} - 3 + 2s} \big(\mathfrak{r}(|\beta| + 3 - 2s) - 3\big)^{-1/\mathfrak{r}} \|\Phi\|_\infty \|\chi\|_\infty \,. \end{aligned} \tag{37}$$

In particular, (when $\mathfrak{r} = 1$, i.e., $\mathfrak{q}^* = \mathfrak{p}$),

$$\|\Phi E_{s,m}(\mathbf{p})^{-1} D^\beta \chi\|_{\mathcal{B}_{\mathfrak{p}}} \leq c_s \frac{\beta!\, d^{2s}}{|\beta| - 2s} \Big(\frac{8}{d}\Big)^{|\beta|} \|\Phi\|_\infty \|\chi\|_\infty \,, \tag{38}$$

for all $\beta \in \mathbb{N}_0^3$ with $|\beta| > 1$.

The operator $\widetilde{E}_{1/2,0}(\mathbf{p})^{-1} := (|\mathbf{p}| + 1)^{-1}$ satisfies the estimates (37) and (38) with $s = 1/2$.

For a proof, see [1, Lemma C.2], which is for $s = 1/2, m > 0$. This proof works, mutatis mutandis, also for general $E_{s,m}(\mathbf{p})^{-1}$, noticing that a formula similar to [1, (C.5)] holds for all $s \in (0,1)$. For $\widetilde{E}_{1/2,0}(\mathbf{p})^{-1}$, one uses

$$\widetilde{E}_{1/2,0}(\mathbf{p})^{-1}(\mathbf{x}, \mathbf{y}) = \frac{1}{\pi^2} \int_0^\infty \frac{\mathrm{e}^{-t|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^2} \frac{t\, dt}{(t^2+1)^2} \,, \tag{39}$$

which follows from $(x+1)^{-1} = \int_0^\infty \mathrm{e}^{-tx} e^{-t}\, dt$, and the explicit expression for the heat kernel of $|\mathbf{p}|$ (see [4, Section 7.11]). Also, one needs

the estimate

$$\Big|\partial_{\mathbf{x}}^{\beta}\frac{\mathrm{e}^{-t|\mathbf{x}|}}{|\mathbf{x}|^2}\Big| \le \frac{2\beta!}{|\mathbf{x}|^2}\Big(\frac{8}{|\mathbf{x}|}\Big)^{|\beta|}\mathrm{e}^{-t|\mathbf{x}|/2} \quad \text{for all } t>0, \mathbf{x}\in\mathbb{R}^3\setminus\{0\}, \beta\in\mathbb{N}_0^3\,, \tag{40}$$

which follows as in [1, Lemma C.3 (C.9)].

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